Power substitution in quasianalytic Carleman classes

Lev Buhovsky[1], *Avner Kiro*[2] *and Sasha Sodin*[3]

Abstract Consider an equation of the form $f(x) = g(x^k)$, where $k > 1$ is an integer and $f(x)$ is a function in a given Carleman class of smooth functions. For each k, we construct a non-homogeneous Carleman-type class which contains all the smooth solutions $g(x)$ to such equations. We prove that if the original Carleman class is quasianalytic, then so is the new class. The results admit an extension to multivariate functions.

1 Introduction

In this text, we consider power substitutions in Carleman classes, i.e. equations of the form $g(x^k) = f(x)$, where $k > 1$ is an integer and f is a given function in a quasianalytic Carleman class C^M (see Definition 1). Our motivation to study power substitutions in Carleman classes mainly comes from [2]. There, it was shown, under certain conditions, that if $F(x, y)$ belong to a quasianalytic Carleman class $C^M(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$ (see Definition 3) and the equation $F(x, y) = 0$ admits a C^∞ solution $y = h(x)$, then h is the image of a $C^M(\mathbb{R}^{d_1})$ function under finitely many power substitutions and blow-ups. Another source of motivation comes from [3, 11], where normalization algorithms for power series in Carleman classes also require finitely many power substitutions and blow-ups.

The results of [2] imply that smooth solutions g of $f(x) = g(x^k)$ inherit a certain quasianalytic property from the original Carleman class: they are definable in an appropriate o-minimal structure. The combination of our main results (Theorems 1 and 2 below) implies the following more explicit quasianalytic property: g belongs to a quasianalytic class $C^M_{1-1/k}$ (Definition 2) completely characterized in terms of bounds on the derivatives of g.

Definition 1. Let $M = (M_n)_{n\geq 0}$ be a positive sequence and let I be an interval. The *Carleman class* $C^M(I)$ consists of all functions $f \in C^\infty(I)$ such that, for any compact set $K \subset I$, there exist constants $A,\ B > 0$ such that

$$\left| f^{(n)}(x) \right| \leq AB^n M_n, \quad x \in K,\ n \geq 0.$$

A Carleman class $C^M(I)$ is said to be *quasianalytic* if any $f \in C^M(I)$ that has a zero formal Taylor expansion at some $x \in I$ is identically zero.

According to the Denjoy–Carleman theorem (see [5] or [9, §12] for this exact formulation) the class $C^M(I)$ is quasianalytic if and only if

$$\sum_{n\geq 0} \frac{M_n^C}{M_{n+1}^C} = \infty \ , \tag{1}$$

[1] School of Mathematical Sciences, Tel Aviv University, Tel Aviv, 69978, Israel. Email: levbuh@tauex.tau.ac.il.
Supported in part by ERC Starting Grant 757585 and ISF Grant 2026/17.
[2] School of Mathematical Sciences, Tel Aviv University, Tel Aviv, 69978, Israel. Email: avnerkiro@gmail.com.
Supported in part by ERC Advanced Grant 692616, ISF Grant 382/15 and BSF Grant 2012037.
[3] School of Mathematical Sciences, Queen Mary University of London, London E1 4NS, United Kingdom & School of Mathematical Sciences, Tel Aviv University, Tel Aviv, 69978, Israel. Email: a.sodin@qmul.ac.uk. Supported in part by the European Research Council starting grant 639305 (SPECTRUM) and by a Royal Society Wolfson Research Merit Award.

where M^C is the largest log-convex minorant of M, i.e.

$$M_n^C = \min\left\{ M_n, \inf_{j<n<\ell} M_j^{(\ell-n)/(\ell-j)} M_\ell^{(n-j)/(\ell-j)} \right\}.$$

A necessary and sufficient condition for the equality of two Carleman classes was given in [6]. In particular, if the sequence M satisfies $M_n \geq n!$ for any $n \geq 0$, then $C^M(I) = C^{M^C}(I)$.

Given $f \in C^M(I)$ where I is an interval such that $0 \in I$ (possibly as an endpoint), we consider a function g defined on the interval $I^k = \{x^k : x \in I\}$ and satisfying $g(x^k) = f(x)$. It is well known that if the class $C^M(I)$ contains all real analytic functions (i.e. there exists $\delta > 0$ such that $M_n^C \geq \delta^{n+1} n!$ for every $n \geq 0$) in I, then $g \in C^M\left(I^k \setminus (-\varepsilon, \varepsilon)\right)$, for any $\varepsilon > 0$ (see Lemma 3.2 below), but g may be singular at zero. If g happens to be C^∞ near zero, then

$$g^{(n)}(0)/n! = f^{(kn)}/(kn)!\ , \tag{2}$$

as follows (for polynomials) from the Cauchy theorem, and thus there exist constants $A,\ B > 0$ such that

$$\left| g^{(n)}(0) \right| \leq AB^n \frac{M_{kn}}{(kn)!^{1-1/k}}, \quad n \geq 0, \tag{3}$$

It was shown in [2] and [10] that under some regularity conditions on the sequence M, the estimate (3) on the derivatives of g at zero can be extended to the interval I^k. A similar fact, without regularity assumptions, follows from a combination of Theorem 1 and Proposition 2.1. Namely, it follows that $g \in C^{M^{(k)}}(I^k)$, where $M_n^{(k)} = n! \sup_{j \leq nk+1} \frac{M_j}{j!}$. Note that by the formula (2) for $g^{(n)}(0)$ there is no smaller Carleman class that contains g. By the Denjoy–Carleman theorem, the classes $C^{M^{(k)}}$ may fail to be quasianalytic even if the original class C^M is quasianalytic. We will show that in the above case, the function g belongs to a new non-homogeneous class $C_{1-1/k}^M(I^k)$ of smooth functions (defined in Definition 2 below) and that the latter class is quasianalytic.

Results similar to these were first obtained by the second author as a byproduct of the work [8]. In the first version of this paper, available on arXiv under the same address, we applied the elementary method of Bang [1] to relax the regularity assumptions at the expense of relinquishing the precise asymptotics. Here, instead of adapting the arguments from classical quasianalyticity, we employ a reduction to the classical setting, and in this way relax the regularity assumption even further.

2 Results

Definition 2. Let M be a a positive sequence, I be an interval and let $0 \leq a$. The class $C_a^M(I)$ consists of all functions $g \in C^\infty(I)$ such that for any compact set $K \subset I$, there exist constants $A,\ B > 0$ such that

$$\left| g^{(n)}(x) \right| \leq AB^n \frac{M_n}{|x|^{an}}, \quad x \in K \setminus \{0\},\ n \geq 0.$$

Theorem 1. *Let $C^M(I)$ be quasianalytic Carleman class, and let $k > 1$ be an integer. Let $g \in C^\infty(I^k)$, and let $f(x) = g(x^k)$. If $f \in C^M(I)$, then $g \in C_a^M(I^k)$, where $a = 1 - \frac{1}{k}$.*

The next proposition demonstrates that functions in $C_a^M(I)$ with $a < 1$ carry additional, implicit, control on their successive derivatives.

Proposition 2.1. *Let M be positive sequence, and let $k > 1$ be an integer. If $g \in C_a^M(I)$ with $a = 1 - \frac{1}{k}$, then $g \in C^{M^{(k)}}(I)$, where*

$$M_n^{(k)} = n! \sup_{j \leq nk+1} \frac{M_j}{j!}.$$

In the case that g is the smooth solution to a power substitution $g(x^k) = f(x)$ with $f \in C^M(I)$, this additional smoothness was already shown in [2, 10].

Our next result is about quasianalyticity of $C_a^M(I)$ with $a < 1$.

Theorem 2. *Let M be positive sequence, and let $0 \le a < 1$. If M is log–convex or $(M_n/n!)_{n\ge 0}$ is non decreasing, then the class $C_a^M(I)$ is quasianalytic if and only if (1) holds.*

There are multivariate analogues to Theorems 1 and 2. We postpone the discussion of such analogues to the last section.

Finally, the next two examples show that when $a \ge 1$, there are no analogues statements to Proposition 2.1 and Theorem 2, even in the simple analytic case, when $M_n = n!$.

Example 1. Consider the $C^\infty[0,1]$ function g, defined by $g(x) = \exp(-1/x)$ for $0 < x \le 1$ (and $g(0) = 0$). By Cauchy's estimates for the derivatives of analytic functions, we have

$$|g^{(n)}(x)| \le n!\frac{2^n}{x^n} \cdot \max_{|z-x|=\frac{x}{2}} |g(z)| \le n!\frac{2^n}{x^n}.$$

So $g \in C_1^M([0,1])$ with $M_n = n!$, and $g^{(n)}(0) = 0$ for any $n \ge 0$. In particular, there is no analogue to Theorem 2 for $a \ge 1$.

Example 2. Let $(N_n)_{n\ge 0}$ be an arbitrary positive sequence. We argue that there exists a function $g \in C_1^{n!}([0,1])$ such that

$$\liminf_{n\to\infty} \frac{|g^{(n)}(0)|}{N_n} > 0. \tag{4}$$

In particular, the existence of such a function shows that the analogue to Proposition 2.1 in the case $a \ge 1$ does not hold.

The construction of g is done in two steps. First, by Borel's Lemma (see [4, p. 44] or [7, p.16]) there is a 2π periodic and $C^\infty(\mathbb{R})$ function, h, such that $h^{(n)}(0) = N_n$, for any $n \ge 0$. Expanding the function h in a Fourier series, we have

$$h(x) = \sum_{j\in\mathbb{Z}} a_j e^{ijx},$$

where $|a_j| = o(|j|^{-m})$ as $|j| \to \infty$, for any $m > 0$.

Next, put

$$h_+(x) := \sum_{j\ge 0} a_j e^{-jx}, \quad h_-(x) = \sum_{j>0} a_{-j} e^{-jx}.$$

Since $|a_j| = o(|j|^{-m})$ as $|j| \to \infty$, for any $m > 0$, the functions $h_\pm$ belongs to $C^\infty\left(\overline{\mathbb{C}_+}\right) \cap \mathrm{Hol}\left(\mathbb{C}_+\right)$, where $\mathbb{C}_+ := \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$. So, as in the previous example, by Cauchy's estimates for the derivatives of analytic functions, we have $h_\pm \in C_1^M([0,1])$ with $M_n = n!$. Moreover, since $h(x) = h_+(ix) + h_-(-ix)$ for any $x \in \mathbb{R}$, either h_+ or h_- satisfies (4).

3 Theorem 1 and Proposition 2.1

Theorem 1 immediately follows from the next lemma.

Lemma 3.1. *Let M be a positive sequence, $k > 1$ be an integer, and $f \in C^\infty[0,1]$ be a function such that $\max_{[0,1]} |f^{(n)}(x)| \le M_n$ for any $n \ge 0$. Put $g(x) = f(x^{1/k})$. If $g \in C^\infty[0,1]$, then*

$$|g^{(n)}(x)| \le \frac{2^n M_n}{x^{(1-1/k)n}}, \quad n \ge 0.$$

Proof. Fix $n \geq 1$, First we write Taylor expansion to f around the origin with integral remainder:

$$f(x) = \sum_{j=0}^{n-1} \frac{f^{(j)}(0)}{j!} x^j + \frac{1}{(n-1)!} \int_0^x f^{(n)}(t)(x-t)^{n-1} dt.$$

Since g is $C^\infty[0,1]$ function, we have $f^{(j)}(0) = 0$ for j which is not divisible by k. Therefore

$$g(x) = P(x) + \frac{1}{(n-1)!} \int_0^{x^{1/k}} f^{(n)}(t)(x^{1/k} - t)^{n-1} dt, \tag{5}$$

where P is a polynomial of degree at most $\frac{n-1}{k}$. Put

$$F(x,t) = \frac{1}{(n-1)!}(x^{1/k} - t)^{n-1} = \frac{1}{(n-1)!} \sum_{j=0}^{n-1} (-1)^{n-1-j} \binom{n-1}{j} x^{j/k} t^{n-1-j}.$$

Differentiating F n times with respect to the variable x yields

$$\frac{\partial^n}{\partial x^n} F(x,t) = \frac{1}{(n-1)!} \sum_{j=0}^{n-1} \left[\left(\prod_{\ell=0}^{n-1} \left(\frac{j}{k} - \ell \right) \right) (-1)^{n-1-j} \binom{n-1}{j} x^{j/k-n} t^{n-1-j} \right].$$

In particular, for $0 < t < x^{1/k}$,

$$\left| \frac{\partial^n}{\partial x^n} F(x,t) \right| \leq \sum_{j=0}^{n-1} \binom{n-1}{j} x^{\frac{n-1}{k} - n} = 2^{n-1} x^{\frac{n-1}{k} - n}. \tag{6}$$

In addition, the chain rule yields

$$\frac{\partial^\ell}{\partial x^\ell} F(x, x^{1/k}) = \begin{cases} \left(\frac{x^{\frac{1}{k}-1}}{k} \right)^{n-1}, & \ell = n-1 \\ 0, & \ell < n-1. \end{cases}$$

Thus, by differentiating (5) n times, we get

$$g^{(n)}(x) = \int_0^{x^{1/k}} \frac{\partial^n}{\partial x^n} F(x,t) f^{(n)}(t) dt + \frac{\partial^{n-1}}{\partial x^{n-1}} F(x, x^{1/k}) f^{(n)}(x^{\frac{1}{k}}) \frac{x^{\frac{1}{k}-1}}{k}.$$

Finally, using (6) we obtain

$$|g^{(n)}(x)| \leq M_n \left(2^{n-1} x^{n/k-n} + \frac{1}{k^n} x^{n/k-n} \right) \leq \frac{2^n M_n}{x^{(1-1/k)n}}.$$

□

The next lemma yields Proposition 2.1 (by taking $n = k\ell + 1$) and it is also being used in the proof of Theorem 2.

Lemma 3.2. *Let M be a positive sequence such that $(M_n/n!)_n$ is non decreasing, $0 < \sigma < 1$, and $g \in C^\infty[0,1]$ be a function such that $|g^{(n)}(x)| \leq \frac{M_n}{x^{(1-\sigma)n}}$ for any $n \geq 0$ and $x \in (0,1]$. Then for any $0 \leq \ell \leq n$ and $0 \leq x \leq 1$ we have*

$$\left| g^{(\ell)}(x) \right| \leq 2^n \frac{\ell!}{n!} M_n \cdot \begin{cases} \frac{(\ell - \sigma n)^{-1}}{x^{\ell - \sigma n}}, & \ell > \sigma n \\ 1 + \log \frac{1}{x}, & \ell = \sigma n \\ (\sigma n - \ell)^{-1}, & \ell < \sigma n. \end{cases}$$

Proof. The case $\ell = n$ is trivial. Assume that $\ell < n$. Writing Taylor expansion of degree $n - \ell - 1$ to the function $g^{(\ell)}$ around 1 with integral remainder, we get

$$g^{(\ell)}(x) = \sum_{j=0}^{n-\ell-1} \frac{g^{(\ell+j)}(1)}{j!}(x-1)^j + \frac{1}{(n-\ell-1)!}\int_1^x g^{(n)}(t)(t-x)^{n-\ell-1}dt.$$

For $x \in (0, 1]$,

$$\begin{aligned}\left|\int_1^x g^{(n)}(t)(t-x)^{n-\ell-1}dt\right| \le \int_x^1 \left|g^{(n)}(t)\right|(t-x)^{n-\ell-1}dt &\le M_n \int_x^1 (t-x)^{n-\ell-1}t^{-(1-\sigma)n}dt \\ &\le M_n \int_x^1 t^{\sigma n-\ell-1}dt.\end{aligned}$$

Hence

$$\begin{aligned}\left|g^{(\ell)}(x)\right| &\le \sum_{j=0}^{n-\ell-1}\frac{M_{\ell+j}}{j!} + \frac{M_n}{(n-\ell-1)!}\int_x^1 t^{\sigma n-\ell-1}dt \le \sum_{j=0}^{n-\ell-1}\frac{(\ell+j)!}{j!}\frac{M_n}{n!} + \ell!\frac{M_n}{n!}\int_x^1 t^{\sigma n-\ell-1}dt \\ &\le \ell!\frac{M_n}{n!}\left(\sum_{j=0}^{n-\ell-1}\binom{n}{j} + \int_x^1 t^{\sigma n-\ell-1}dt\right) \le 2^n\ell!\frac{M_n}{n!}\left(1 + \int_x^1 t^{\sigma n-\ell-1}dt\right).\end{aligned}$$

Since

$$\int_x^1 t^{\sigma n-\ell-1}dt \le \begin{cases}(\ell-\sigma n)^{-1}x^{\sigma n-\ell}, & \ell > \sigma n \\ \log\frac{1}{x}, & \ell = \sigma n \\ (\sigma n-\ell)^{-1}, & \ell < \sigma n,\end{cases}$$

we obtained the desired bound. □

4 Theorem 2

Lemma 4.1. *Let M be a positive sequence such that $(M_n/n!)_n$ is non decreasing, $1 < k$ be an integer, and $g \in C^\infty(0,1]$ be a function such that $|g^{(n)}(x)| \le \frac{M_n}{x^{(1-1/k)n}}$ for any $n \ge 0$ and $x \in (0,1]$. Put $f(x) = g(x^k)$. Then there exist constants A, $C > 0$ such that*

$$\left|f^{(n)}(x)\right| \le AC^nM_n\begin{cases}1, & k\nmid n \text{ or } n = 0; \\ 1+\log\frac{1}{x}, & k \mid n.\end{cases}$$

Proof. First we argue by induction on n that

$$g^{(n)}(x) = \sum_{\substack{i+j=n \\ 1\le i\le n,\ i(k-1)\ge j}} B_n(i,j)f^{(i)}(x^k)x^{i(k-1)-j} \tag{7}$$

where

$$|B_n(i,j)| \le C^n n^{n-i}.$$

Indeed,

$$\frac{d}{dx}\left(f^{(i)}(x^k)x^{i(k-1)-j}\right) = kf^{(i+1)}(x^k)x^{(i+1)(k-1)-j} + (i(k-1)-j)x^{i(k-1)-(j+1)}f^{(i)}(x^k)$$

implies that

$$B_{n+1}(i,j) = kB_n(i-1,j) + (i(k-1)-j)\,B_n(i,j-1).$$

So making use of the induction hypothesis, we find that

$$|B_{n+1}(i,j)| \le kC^n n^{n+1-i} + (ik-n)C^n n^{n-i} \le C^{n+1}(n+1)^{n+1-i}$$

as claimed.

Using (7), we find that

$$\left|g^{(n)}(x)\right| \le C^n \sum_{n/k\le i\le n} n^{n-i}\left|f^{(i)}(x^k)x^{ik-n}\right|$$

By Lemma 3.2,

$$\left|f^{(i)}(x^k)x^{ik-n}\right| \le 2^n\frac{i!}{n!}M_n\begin{cases}1+\log\frac{1}{x}, & i=\frac{n}{k}\\ 1, & i>\frac{n}{k}.\end{cases}$$

Thus in the case $k \mid n$, we find that

$$\left|g^{(n)}(x)\right| \le (2C)^n M_n \sum_{n/k<i\le n}\frac{n^{n-i}i!}{n!} + \frac{n^{n(1-1/k)}\left(\frac{n}{k}\right)!}{n!}\left(1+\log\frac{1}{x}\right) \le (2C)^n M_n\left(1+\log\frac{1}{x}\right),$$

while in the case $k \nmid n$, we have

$$\left|g^{(n)}(x)\right| \le (2C)^n M_n \sum_{n/k<i\le n}\frac{n^{n-i}i!}{n!} \le (2C)^n M_n.$$

□

Proof of Theorem 2. Without loss of generality we assume that $I \subseteq [0,\infty)$. Assume first that $\sum_{n\ge0}\frac{M_n^C}{M_{n+1}^C} = \infty$. Let $g \in C_a^M(I)$ with $g^{(n)}(x_0) \equiv 0$. We need to show that $g \equiv 0$. If $x_0 \ne 0$, then $g \in C^M(I\setminus\{0\})$, in particular by the Denjoy–Carleman Theorem $g \equiv 0$. On the other hand, if $x_0 = 0$, then by replacing $g(x)$ with $g(x/C)$ with sufficiently large $C>0$, there is no loss of generality with assuming that $[0,1]\subseteq I$. Let $k\in\mathbb{N}$ such that $1-\frac{1}{k} > a$.

If M is log-convex, denote by $\widehat{M}$ the log-convex sequence defined by

$$\widehat{M}_0 = 0, \quad \frac{\widehat{M}_{n-1}}{\widehat{M}_n} = \min\left\{\frac{M_{n-1}}{M_n}\ ,\ \frac{1}{n}\right\}.$$

Note that $\widehat{M_n} \ge M_n$ for any $n \ge 0$. Moreover,

$$\frac{\widehat{M}_n}{n!} = \prod_{j=1}^{n}\max\left\{\frac{jM_j}{M_{j-1}}\ ,\ 1\right\}$$

, so the sequence $(\widehat{M}_n/n!)_{n\ge0}$ is non-decreasing. By the condensation test for convergence,

$$\sum_{n\ge0}2^n\frac{M_{2^n-1}}{M_{2^n}} = \infty \quad\Rightarrow\quad \sum_{n\ge0}\min\left\{2^n\frac{M_{2^n-1}}{M_{2^n}}\ ,\ 1\right\} = \infty \quad\Rightarrow \sum_{n\ge0}\frac{\widehat{M}_n}{\widehat{M}_{n+1}} = \infty.$$

On the other hand, if the sequence $(\widehat{M}_n/n!)_{n\ge0}$ is non decreasing, then we put $\widehat{M} = M$.

In both cases, the function g belongs to $C^{\widehat{M}}_{1-1/k}([0,1])$, and $\widehat{M}$ satisfies the assumptions of Lemma 4.1 and $\sum_{n\geq 0}\frac{\widehat{M}_n^C}{\widehat{M}_{n+1}^C} = \infty$. Consider the function $h(x) := \int_0^x g(y^k)dy$. By Lemma 4.1, $h \in C^{\widehat{M}}([0,1])$ and since $g^{(n)}(0) \equiv 0$, then also $h^{(n)}(0) \equiv 0$. By the Denjoy–Carleman Theorem, $h \equiv 0$ and therefore also $g \equiv 0$, as claimed.

Next, if $\sum_{n\geq 0}\frac{M_n^C}{M_{n+1}^C} < \infty$, then by Denjoy–Carleman Theorem, the class $C^M(I)$ is not quasianalytic. Thus there exists a non-zero function $g \in C^M(I)$ which is compactly supported in $I \setminus \{0\}$. Such a function g belongs to the class $C^M_a(I)$, therefore the latter is not quasianalytic. □

5 The multivariate case

5.1 Carleman classes. Here we will use standard multiindex notation: If $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{Z}_+^d$ and $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$, we write $|\alpha| := \alpha_1 + \cdots + \alpha_d$, and $\dfrac{\partial^{|\alpha|}}{\partial x^\alpha} := \dfrac{\partial^{\alpha_1+\cdots+\alpha_d}}{\partial x_1^{\alpha_1}\cdots \partial x_d^{\alpha_d}}$.

Definition 3. Let M be a sequence of positive numbers which is logarithmically convex, and let Ω be a connected set in $\mathbb{R}^d$. *The Carleman class* $C^M(\Omega)$ consists of all functions $f \in C^\infty(\Omega)$ such that for any compact $K \subset \Omega$, there exists $A,\ B > 0$ such that

$$\left|\frac{\partial^{|\alpha|}}{\partial x^\alpha} f(x)\right| \leq AB^{|\alpha|}M_{|\alpha|}, \quad x \in \Omega$$

and for any multiindex $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{Z}_+^d$.

Definition 4. Let M be a regular sequence of positive numbers, Ω be a connected set in $\mathbb{R}^d$, and let $a = (a_1, \cdots, a_d) \in \mathbb{R}_+^d$. The class $C^M_a(\Omega)$ consists of all functions $g \in C^\infty(\Omega)$ such that for any compact $K \subset \Omega$, there exists $A,\ B > 0$ such that

$$\left|\frac{\partial^{|\alpha|}}{\partial x^\alpha} g(x)\right| \leq AB^{|\alpha|}\frac{M_{|\alpha|}}{|x_1|^{a_1\cdot\alpha_1}|x_2|^{a_2\cdot\alpha_2}\cdots|x_d|^{a_d\cdot\alpha_d}}$$

for any $x \in K$ and any multiindex $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{Z}_+^d$.

Definition 5. A set $\Omega \subset \mathbb{R}^d$ is called star-shaped (with respect to the origin) if for any $x \in \Omega$, $\{tx : t \in [0,1]\} \subset \Omega$.

5.2 Results. The next result is the muliivariate version of Theorem 1, and follows from the latter by induction over the dimension.

Theorem 3. *Let $C^M([0,1]^d)$ be a quasianalytic Carleman class. For $k \in \mathbb{N}^d$, denote by $y_k : \mathbb{R}^d \to \mathbb{R}^d$ the map defined by $y_k(x) = (x_1^{k_1}, \cdots, x_d^{k_d})$. If $g \in C^\infty([0,1]^d)$ is such that $f = g \circ y_k \in C^M([0,1]^d)$, then $g \in C^M_a([0,1]^d)$ with $a = (\frac{k_1-1}{k_1}, \cdots, \frac{k_d-1}{k_d})$.*

The next result is the multivariate version of Theorem 2, and it follows form it by restricting functions from $C^M_a(\Omega)$ to lines.

Theorem 4. *Let M be positive sequence, $a \in [0,1)^d$, and and $\Omega \subset \mathbb{R}^d$ be star-shaped. If M is log–convex or $(M_n/n!)_{n\geq 0}$ is non-decreasing, then the class $C^M_a([0,1]^d)$ is quasianalytic if and only if (1) holds.*

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